

## Mean Value Analysis of Closed Queueing Networks with Erlang Service Time Distributions

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Abstract — Zusammenfassung

**Mean Value Analysis of Closed Queueing Networks with Erlang Service Time Distributions.** The classical mean value analysis approach is extended to single class closed queueing networks containing Erlang service time distributions and FCFS scheduling disciplines. A new formula for the mean residence time of jobs is derived. Each iteration provides self-checks for validity, and is repeated whenever invalid results are detected. On the average, the solutions obtained vary by less than five percent from their respective simulation results.

**Mittelwertanalyse für geschlossene Bedienungssysteme mit Erlang-verteilter Bedienzeiten.** Die klassische Mittelwertanalyse wird auf geschlossene Warteschlangennetze mit einer Auftragsklasse, Erlang-verteilter Bedienzeiten und FCFS Zuteilungsmechanismen erweitert. Eine neue Formel für die mittlere Verweilzeit wird hergeleitet. Jede Iteration sieht Gültigkeitskontrollen vor und wird bei Feststellung ungültiger Ergebnisse wiederholt. Die so gewonnenen Lösungen unterscheiden sich im Durchschnitt um weniger als 5% von entsprechenden Simulationsergebnissen.

### 1. Introduction

Queueing networks have enjoyed great popularity as models of computer systems and communication systems since the early 1970's. This is primarily due to their ability to model multiple independent resources and the sequential use of these resources by different jobs. The basic results of queueing theory were presented by Jackson [JACK63], Gordon/Newell [GORD67] and Buzen [BUZE73]. They demonstrated that solutions to open and closed queueing networks with exponentially distributed arrival and service times implementing a First-Come-First-Served queueing discipline have a *product form*. A product form implies that all stations have equilibrium state probabilities consisting of factors representing the individual stations within the network. The resulting implication is that the individual stations behave as if they were separate queueing systems. Baskett, Chandy, Muntz, and Palacios [BCMP75] extended these results to obtain product form solutions for open, closed, and mixed queueing networks with multiple job classes, non-

exponential service time distributions, and different queueing disciplines. Four different types of queueing disciplines were recognized that yield a product form solution. These four are: Type I  $\rightarrow$  First-Come-First-Served with exponential service time distributions; Type II  $\rightarrow$  Round-Robin Processor-Sharing (RR-PS); Type III  $\rightarrow$  Infinite-Servers (IS); and Type IV  $\rightarrow$  Last-Come-First-Served-Preemptive-Resume (LCFS-PR). The latter three queueing disciplines can use any general service time distribution having rational Laplace transform.

Mean value analysis has enjoyed widespread popularity during recent years as an exact technique for providing solutions to product form closed queueing networks. The basic concept of mean value analysis is the application of an iterative procedure to calculate mean residence time, system throughput and the mean number of jobs. A number of studies about mean value analysis have been published in the last few years. These include the classical papers by Reiser, Lavenberg [REIS80] and by Reiser [REIS81] as well as numerous contributions by other scholars [ZAHO81, NEUS81, KRZE84, LAZO84, AKBO87]. The principle advantage to mean value analysis lies in its ability to compute the performance measures without calculating the normalization constants.

Despite their popularity, several drawbacks do exist with product form networks. Probably the most significant of these is the assumptions that must be made when designing the system model. It is these assumptions that allow us to fit a given model to the format required for obtaining performance measures using product form network algorithms. Nevertheless, not all queueing network models will conform to one of the classes covered by product form algorithms. A queueing model containing even a single station not meeting one of the above mentioned four basic types does not have a product form solution. This introduces problems when one considers the fact that service time distributions tend to demonstrate a high variance at CPUs (hyperexponential) and low variances at the I/O devices (Erlang). Furthermore, incorrectly assuming an exponential service time distribution can introduce significant errors into the results of performance evaluation for actual systems.

There are three standard methods for obtaining performance measures of a non-product form network. First, assumptions can be simplified to the point where one of the product form types now fits the model. The model can then be analyzed using exact analysis. However, this procedure leads to results that are exact but not applicable to the actual system in question.

Secondly, numerical methods, as proposed by Stewart [STEW80] can be used. These methods are based on the computation of state probabilities using the transition rate matrix. Due to the rapid growth of the transition rate matrix with the increase in number of stations and/or jobs, the method is inappropriate for all but small queueing network models.

Third and finally, approximation methods exist for solutions to queueing networks not fitting one of the types required for BCMP networks and its extensions. A large variety of classical approximation methods exist for dealing with distributions and/or scheduling disciplines not containing product form [COUR77, CHAN75, GELI75, KOBA74, KOUV86, MARI80, WALSH85].

In this work we propose an algorithm, based on the theory of mean value analysis. The classical mean value analysis is modified to reflect the actions of a Erlang distributed service time when dealing with a First-Come-First-Served (FCFS) service discipline. The algorithm is conceptually and computationally simple, yet normally provides results within five percent of the actual values.

## 2. Extended Mean Value Analysis

We consider closed queueing network models having the following characteristics. All models contain  $N$  single-server stations. Service times for these jobs are Erlang distributed with a mean value of  $1/\mu_i$  for  $i=1, \dots, N$  and a coefficient of variation  $0 < c_i \leq 1$ . The service discipline is First-Come-First-Served (FCFS) at all stations. There are  $K$  jobs in the system. Jobs completing at station  $i$  proceed to station  $j$  with probability  $p_{ij}$ .

The above model does not have a product form solution, as the stations contain an Erlang service time distribution and a First-Come-First-Served (FCFS) queueing discipline. In other words, the existing product form network algorithms cannot be utilized in obtaining accurate results for the performance measures.

The classical mean value analysis of Reiser Lavenberg [REIS80] developed from two major theorems: The Arrival Instant Distribution Theorem by Sevciik-Mitrani [SEVC81] and Little's Law. The first theorem allows the development of the formula for mean residence time of a job in the  $i$ -th station.

$$t_i(k) = \frac{1}{\mu_i} [1 + \bar{k}_i(k-1)] \quad (1)$$

where  $\bar{k}_i(k-1)$  is the mean number of jobs in the  $i$ -th station assuming there are  $(k-1)$  jobs in the entire network. Informally, the above formula states that the mean residence time of a job entering the  $i$ -th station is given by its own mean service time plus the mean service time of all jobs already queued or in service at that station.

The second theorem allows the computation of the network throughput.

$$\lambda(k) = \sum_{j=1}^N c_j t_j(k) \quad (2)$$

where

$p_{ij}$  is the probability that a job in the  $i$ -th station proceeds to the  $j$ -th station.

The mean number of jobs at the  $i$ -th station can also be derived from Little's Law.

$$k_i(k) = c_i \lambda(k) t_i(k) \quad (3)$$

$k_i(0) = 0$  is assumed as the initial value for the iteration. The iteration terminates when the total number of jobs in the network is reached.

As stated previously, mean value analysis can only be applied on product form networks. In particular, the Arrival Instant Distribution Theorem only holds for product form networks. In order to develop the mean residence time formula for general networks, we use the following observation. The mean residence time for a job entering the station is computed by its own mean service time, the mean residence time for all jobs waiting in the queue, and the remaining mean service time for the job currently in service  $RST_i$ . We note that  $\bar{q}_i(k-1)$  is the mean number of jobs enqueued at the  $i$ -th station given that there are  $(k-1)$  jobs in the network.

$$\bar{r}_i(k) = \frac{1}{\mu_i} [1 + \bar{q}_i(k-1)] + RST_i \quad (4)$$

When computing the mean residence time for all jobs waiting in the queue, we find that  $\bar{q}_i(k-1)$  is equivalent to  $\bar{k}_i(k-1)$  if we remove from consideration the job currently receiving service at that station,  $\rho_i(k-1)$ . Equation (4) for mean residence time of a job entering station  $i$  now becomes:

$$\bar{r}_i(k) = \frac{1}{\mu_i} [1 + \bar{k}_i(k-1) - \rho_i(k-1)] + RST_i \quad (5)$$

We must now determine the value of  $RST_i$ , using the following well-known theorem [SAUE81].

**Theorem:** *The remaining mean service time of a job in service as viewed by a new job entering the M/G/1-FCFS system is defined to be:*

$$RST = \frac{\lambda}{2} E[S^2] = \frac{\rho}{2} \frac{E[S^2]}{E[S]} \quad (6)$$

where

$E[S]$  is the first moment of the service time.

$E[S^2]$  is the second moment of the service time.

*Proof:*

The variation coefficient is given by:

$$c = \frac{\sqrt{\text{var}[S]}}{E[S]}$$

where the variance is defined as  $\text{var}[S] = E[S^2] - E[S]^2$ .

The above two equations can be used to obtain

$$\begin{aligned} c^2 E[S]^2 &= E[S^2] - E[S]^2 \\ (c^2 + 1)E[S] &= \frac{E[S^2]}{E[S]} \end{aligned} \quad (7)$$

By substituting equation (7) into equation (6) we obtain

$$RST = \frac{\rho}{2} (c^2 + 1) E[S]$$

since the mean service time is assumed to be  $1/\mu = E[S]$ . It therefore follows that:

$$RS_i = \frac{\rho}{2} \frac{1}{\mu} (c_i^2 + 1). \quad (8)$$

This completes the proof.  $\square$

We now insert this value for remaining service time into equation (5), yielding the following equation for mean residence time.

$$\bar{t}_i(k) = \frac{1}{\mu_i} [1 + \bar{k}_i(k-1) - \rho_i(k-1)] + \frac{\rho_i(k-1)}{2} \frac{1}{\mu_i} (c_i^2 + 1)$$

This equation can be re-written as:

$$\bar{t}_i(k) = \frac{1}{\mu_i} [1 + \bar{k}_i(k-1)] - \frac{1}{\mu_i} \rho_i(k-1) + \frac{\rho_i(k-1)}{2} \frac{1}{\mu_i} (c_i^2 + 1)$$

and

$$\bar{t}_i(k) = \frac{1}{\mu_i} [1 + k_i(k-1)] - \frac{1}{\mu_i} \rho_i(k-1) \left[ \frac{1 - c_i^2 + 1}{2} \right]$$

and finally

$$\bar{t}_i(k) = \frac{1}{\mu_i} [1 + \bar{k}_i(k-1)] + \frac{1}{\mu_i} \rho_i(k-1) \left[ \frac{c_i^2 - 1}{2} \right]. \quad (9)$$

Since  $\rho_i(k-1) = x_i \lambda(k-1)$ , we can now obtain the formula for mean residence time for a general network utilizing a First-Come-First-Served queueing discipline.

$$\bar{t}_i(k) = \frac{1}{\mu_i} [1 + \bar{k}_i(k-1) + 0.5 x_i \lambda(k-1) (c_i^2 - 1)] \quad (10)$$

where

$c_i$  represents the variation coefficient of the  $i$ -th station's service time distribution.

$x_i = \frac{e_i}{\mu_i}$  represents the relative utilization of station  $i$ .

It should be pointed out that the new equation (10) reduces to classical mean value analysis, equation (1), when the coefficient of variation  $c_i$  has the value one.

Since Little's law is valid for all queueing systems, we can use equations (2) and (3) for the derivation of system throughput and mean number of jobs.

Equations (10), (2), and (3) allow us to set up a recursive evaluation of the mean residence times, throughputs, and mean number of jobs present at all stations for a general closed queueing network model. Initial values are shown below.

$$k_i(0) = 0, \quad \lambda(0) = 0.$$

As with the classical mean value analysis, our algorithm terminates when the total number of jobs in the network is reached.

The mean value analysis algorithm revisions mentioned this far are similar to that proposed by [REIS78], [SAUER81] and [BOND86]. In this intermediate form,

however, the approximations do not provide accurate results. In fact, the algorithm demonstrates an unstable behavior. The primary cause for inaccuracies is the violation of the stability condition:

$$\rho_i(k) = \lambda_i(k) / \mu_i < 1.$$

Simply stated, this intermediate form of the algorithm permits a station to take on a utilization value greater than 100% during the iterations. This is an obvious violation of the queueing theory.

Our algorithm recognizes this impossibility. Whenever a stability condition violation occurs, a new utilization value is assigned to that station:

$$\rho_i(k) = 0.99999. \quad (11)$$

Introducing equation (11) modifies the balance of the network. We then now compute a new throughput value for the current iteration.

$$\lambda_i(k) = \rho_i(k) \mu_i \quad (12)$$

Since  $\lambda_i(k) = e_i \lambda(k)$  it follows from equations (11) and (12) that

$$\lambda(k) = \frac{\mu_i}{e_i} 0.99999. \quad (13)$$

Given this new throughput value, we can now compute the correct utilization  $\rho_i$  and mean number of jobs  $\bar{k}_i(k)$  for all stations  $j = 1, \dots, N$  using equation (3). This new value for throughput will reduce the mean number of jobs  $\bar{k}_i$  in each station. As a result, the total number of jobs present will be smaller than  $k$ , the number of jobs in that iteration.

$$\sum_{i=1}^N \bar{k}_i(k) < k$$

We determine the number of remaining jobs by:

$$\bar{k}_{rem}(k) = k - \sum_{i=1}^N \bar{k}_i(k). \quad (14)$$

The value  $\bar{k}_{rem}$  is placed in the totally utilized station:

$$\bar{k}_i(k) = \bar{k}_i(k) + \bar{k}_{rem}(k) \quad (15)$$

where  $i$  is the totally utilized station. Since we now have a new total throughput and mean number of jobs value for the totally utilized station, we recompute an accurate mean residence time  $\bar{t}_i(k)$ .

$$\bar{t}_i(k) = \frac{\bar{k}_i(k)}{e_i \lambda(k)} \quad (16)$$

It should be noted that this last step, equation (16), will improve the accuracy of the results for this iteration, but will not effect future iterations of the algorithm, as the value for mean residence time is not used in the next iteration.

The following summarizes the Extended Mean Value Analysis Algorithm.

## Extended Mean Value Analysis Algorithm — EMVA

```

begin
  for all stations  $i=1$  to  $N$  do
    •  $k_i(0)=0$ 
    •  $\lambda_i(0)=0$ 
    • compute  $v_i$ , the mean number of visits by a job to station  $i$ 
    • compute  $x_i$ , the relative utilization of station  $i$ 
  end

  for all jobs in the network  $k=1$  to  $K$  do
    for all stations  $i=1$  to  $N$  do
      • compute mean residence time  $\bar{r}_i(k)$  using equation (10)
    end
    • compute system throughput  $\lambda(k)$  using equation (2)
    for all stations  $i=1$  to  $N$  do
      • compute mean number of jobs  $k_i(k)$  using equation (3)
      • compute utilization of each station using  $x_i$  and  $\lambda(k)$ 
    end

    if there exists a  $\rho_i > 0.99999$  then
      • set  $\rho_i = 0.99999$ 
      • recompute throughput value using equation (13)
      for all stations  $i=1$  to  $N$  do
        • recompute mean number of jobs from equation (3)
      end
      • determine the number of remaining jobs  $k_{rem}$  from equation (14)
      • add  $k_{rem}$  to the mean number of jobs  $k_i$  for the totally utilized station, equation (15)
      • correct the mean residence time  $\bar{r}_i$  for the totally utilized station, equation (16)
    endif
  end {for  $k=1$  to  $K$ }
end

```

In the following numerical example, the general flow of the algorithm is outlined.

## 3. Example

Consider a closed central server model with  $N=3$  (1 CPU, 2 I.O's) stations and  $K=3$  jobs. The CPU service time is Erlang distributed with mean value  $1/\mu_1=2$  and variation coefficient  $c_1=0.707$ . The I.O service times are also Erlang distributed with mean values  $1/\mu_2=0.5$  and  $1/\mu_3=1$  and with the same coefficient of variation  $c_i=0.707$  (for  $i=2,3$ ). The transition probabilities are given by  $p_{12}=p_{13}=0.5$ .

The first iteration provides us the following results:

$$\begin{array}{lll}
 \bar{r}_1(1)=2 & & k_1(1)=0.727 \\
 \bar{r}_2(1)=0.25 & \lambda(1)=0.364 & k_2(1)=0.091 \\
 \bar{r}_3(1)=0.5 & & k_3(1)=0.182
 \end{array}$$

We repeat the iterations for  $k=2$  and  $k=3$  and obtain the following results

$$\begin{array}{lll}
 \bar{r}_1(3)=4.637 & & k_1(3)=2.517 \\
 \bar{r}_2(3)=0.276 & \lambda(3)=0.543 & k_2(3)=0.15 \\
 \bar{r}_3(3)=0.613 & & k_3(3)=0.333
 \end{array}$$

and

$$\rho_1(3) = 1.085$$

$$\rho_2(3) = 0.136$$

$$\rho_3(3) = 0.271$$

Obviously, the first station is totally utilized, so we set

$$\rho_1(3) = 0.999$$

and compute the new throughput value

$$\lambda(3) = \frac{\mu_1}{e_1}$$

We recompute the performance measure as follows:

$$\rho_1(3) = 0.99$$

$$\bar{k}_1(3) = 2.319$$

$$\rho_2(3) = 0.125$$

$$\bar{k}_2(3) = 0.138$$

$$\rho_3(3) = 0.25$$

$$\bar{k}_3(3) = 0.306$$

The sum of  $k_i(3)$  values does not provide 3, the total number of jobs in the network. We determine the number of "lost" jobs by equation (5):

$$\bar{k}_{\text{rem}} = 0.237$$

and obtain by adding this value to the totally utilized station the following

$$\bar{k}_1(3) = 2.556$$

The final results are listed in the following table:

	EMVA	Exact	Deviation
$\rho_1$	0.999	0.995	0
$\rho_2$	0.125	0.124	0
$\rho_3$	0.25	0.249	0
$\bar{k}_1$	2.556	2.587	8
$\bar{k}_2$	0.138	0.132	4.5
$\bar{k}_3$	0.306	0.281	8.9

The deviations are computed from the following formula:

$$\text{Deviation} = \left\{ \frac{| \text{Exact} - \text{EMVA} |}{\text{Exact}} \right\} \cdot 100$$

#### 4. Evaluation

We have derived a formula for the mean residence time in a closed queuing network with an Erlang service time distribution and First-Come-First-Served (FCFS) queuing discipline. These networks do not have a product form, and thus exact solutions are not realizable. The technique provides approximate results (within five percent on the average) for such networks.



The algorithm presented appears to have the advantage that it incorporates the standard mean value analysis as a subset of itself.

Many different networks containing two to ten stations were analyzed by the proposed algorithm, with the number of jobs ranging from ten to 150 in each network. In all cases, we observed variations of less than fifteen percent from the simulation results. The vast majority of these showed variations beneath five percent. It is clearly evident that this technique is capable of accurate modeling. It should be noted that all instances in which our algorithm showed a relatively high deviation from the actual results (over 10 percent), the numbers involved were quite small. In such cases the relative error might appear large even though the difference in the two numbers is insignificant. Under these circumstances, the relative error cannot be considered a reliable indicator of the accuracy of our algorithm.

A detailed comparison between EMVA and other existing algorithms has also been conducted. As mentioned in the introduction, several approximate methods have been proposed in the last decade for closed queueing networks with general service time distributions. Our study has shown that the Method of Marie [MARI80] is the most reliable of these methods, if for networks in which all stations contain a coefficient of variation  $c$  greater than 0.5 ( $c \geq .5$ ). This fact eliminates the analysis of networks containing any station with an Erlang service time distribution of four or more phases by the Method of Marie.

The algorithm in its current form is unable to handle hyper-exponential service time distribution. The accuracy of its results to appear to deteriorate in those cases. The space complexity of the algorithm is the same as that of the classical mean value analysis. That is,  $O[N(K+1)]$ , where  $N$  represents the number of stations in the network and  $K$  the number of jobs. Analyzing the time complexity of EMVA shows it to be on the same order as that of the classical mean value analysis. Both the time and space complexity of EMVA show it to be superior to the requirements of other approximate methods.

## 5. Appendix

The following is a list of all models tested in the simulations. Each model contains a brief introduction, a formal listing of parameters, and the comparison between extended mean value analysis and the simulation results. The deviations are computed by the following formula:

$$Dev = \left\{ \frac{|simulation \text{ (or exact)} - analytical \text{ method}|}{simulation \text{ (or exact)}} \right\} \cdot 100$$

### Model 1

This model is composed of three stations representing a CPU and two I/O devices. The model is a combination of exponential and Erlang service time distributions. Five jobs were tested so that exact analysis could be executed. The Method of Marie

and also the Extended Product Form (EPF) technique is included for comparison. The MVA solutions are obtained by assuming the coefficient of variation value  $c_i = 1$ .

Station	Number	$\mu_i$	$c_i$	$p_{i1}$	$p_{i2}$	$p_{i3}$
CPU	1	1	1	0	0.5	0.5
I/O 1	2	0.5	0.7071	1	0	0
I/O 2	3	1	0.7071	1	0	0

Mean Residence Time - $\bar{r}(S)$								
Exact	EMVA	Dev.	Marie	Dev.	EPF	Dev.	MVA	Dev.
2.643	2.861	8.2%	2.674	1.1%	2.685	1.6%	2.721	3%
5.250	4.792	8.7%	5.215	0.6%	5.227	0.4%	5.441	4%
1.448	1.445	0%	1.432	1.1%	1.449	0%	1.558	7.6%

Mean Number of Jobs - $\bar{k}(S)$								
Exact	EMVA	Dev.	Marie	Dev.	EPF	Dev.	MVA	Dev.
2.205	2.392	8.4%	2.229	1.1%	2.229	1.1%	2.187	0.8%
2.190	2.003	8.5%	2.174	0.7%	2.162	1.3%	2.187	0.1%
0.604	0.604	0%	0.596	0%	0.608	0%	0.626	3.6%

Utilization - $\rho(S)$								
Exact	EMVA	Dev.	Marie	Dev.	EPF	Dev.	MVA	Dev.
0.834	0.836	0%	0.833	0%	0.830	0%	0.803	3.7%
0.834	0.836	0%	0.833	0%	0.827	0.8%	0.803	3.7%
0.417	0.418	0%	0.416	0%	0.419	0%	0.401	3.8%

Throughput - $\lambda(S)$								
Exact	EMVA	Dev.	Marie	Dev.	EPF	Dev.	MVA	Dev.
0.834	0.836	0%	0.833	0%	0.830	0%	0.803	3.7%
0.417	0.418	0%	0.416	0%	0.413	0%	0.401	3.8%
0.417	0.418	0%	0.416	0%	0.413	0%	0.401	3.8%

**Model 2**

This model tests a network composed of Erlang service time distribution stations only. There were  $K = 20$  jobs in this example. The Method of Marie cannot analyze this model. We compare the EMVA results with simulation, EPF and MVA.

Station	Mean Service Rate	Coefficient of Variation	Routing Probabilities
1	$\mu_1 = 4.0$	0.7071 (2 phases)	$p_{12} = 0.45, p_{13} = 0.40, p_{14} = 0.15$
2	$\mu_2 = 0.75$	0.5773 (3 phases)	$p_{21} = 1.0$
3	$\mu_3 = 0.75$	0.5000 (4 phases)	$p_{31} = 1$
4	$\mu_4 = 0.4$	0.4772 (5 phases)	$p_{41} = 1$

Mean Residence Time -  $\bar{t}(20)$

Simulation	EMVA	Dev.	EPF	Dev.	MVA	Dev.
0.3195	0.3839	20.16%	0.380	19%	0.422	32.14%
18.147	17.9342	1.17%	17.345	4.4%	16.382	9.72%
7.0045	6.9682	0.51%	7.664	9.4%	8.617	23.02%
4.9795	5.0555	1.53%	5.499	11%	6.403	28.59%

Mean Number of Jobs -  $\bar{k}(20)$

Simulation	EMVA	Dev.	EPF	Dev.	MVA	Dev.
0.5313	0.6398	20.42%	0.627	18%	0.6920	30.25%
13.5629	13.4507	0.82%	12.91	4.8%	12.0837	10.91%
4.6610	4.6455	0.33%	5.095	9.3%	5.6498	21.21%
1.2446	1.2638	1.54%	1.365	9.7%	1.5743	26.49%

Utilization -  $\rho(20)$

Simulation	EMVA	Dev.	Marie	Dev.	MVA	Dev.
0.4156	0.4219	1.12%	0.412	0.8%	0.4097	1.42%
0.9961	0.9999	0.38%	0.992	0.4%	0.9834	1.27%
0.8863	0.9000	1.55%	0.886	0%	0.8741	1.38%
0.6240	0.6328	1.41%	0.621	1.4%	0.6146	1.51%

Throughput - $\lambda(20)$						
Simulation	EMVA	Dev.	EPF	Dev.	MVA	Deg.
1.6627	1.6666	0.23%	1.650	0.7%	1.6391	1.42%
0.7474	0.7499	0.33%	0.744	0.4%	0.7376	1.31%
0.6553	0.6666	1.72%	0.664	1.3%	0.6556	0.05%
0.2499	0.2499	0.00%	0.248	0.7%	0.2458	1.64%

### Model 3

This model tests a network similar to the example in which a stress test is conducted on Erlang distribution stations. Again, fifty jobs were tested. The EMVA results are compared with simulation and MVA results.

Station	Mean Service Rate	Coefficient of Variation	Routing Probabilities
1	$\mu_1 = 1.0$	1.0000 (exponential)	$p_{12} = 0.7, p_{13} = 0.3$
2	$\mu_2 = 9.0$	0.316228 (10 phases)	$p_{23} = 0.5, p_{24} = 0.5$
3	$\mu_3 = 2.0$	0.223607 (20 phases)	$p_{31} = 0.6, p_{34} = 0.4$
4	$\mu_4 = 0.75$	0.200000 (25 phases)	$p_{41} = 1$

Mean Residence Time - $\bar{r}(50)$				
Simulation	EMVA	Dev.	MVA	Dev.
46.7597	46.8557	0.21%	45.0788	3.59%
0.1163	0.1162	0.09%	0.1205	3.61%
0.6250	0.6263	0.21%	0.7407	18.51%
4.3763	4.3537	0.52%	7.1404	63.16%

Mean Number of Jobs - $\bar{k}(50)$				
Simulation	EMVA	Dev.	MVA	Dev.
46.8377	46.8556	0.04%	45.0785	3.76%
0.0815	0.0814	0.12%	0.0843	3.44%
0.4067	0.4071	0.10%	0.4847	19.18%
2.6740	2.6557	0.68%	4.3556	62.89%

Utilization - $\rho(50)$				
Simulation	EMVA	Dev.	MVA	Dev.
1.0000	0.9999	0.01%	0.9999	0.01%
0.0778	0.0777	0.13%	0.0777	0.13%
0.3254	0.3249	0.15%	0.3249	0.15%
0.8148	0.8133	0.18%	0.8124	0.29%

Throughput - $\lambda(50)$				
Simulation	EMVA	Dev.	MVA	Dev.
1.0017	0.9999	0.18%	0.9999	0.18%
0.7006	0.6999	0.10%	0.6993	0.19%
0.6507	0.6499	0.12%	0.6499	0.12%
0.6111	0.6099	0.20%	0.6093	0.29%

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